Mathematics: The Core Course for A-level; L. Bostock and S. Chandler

Chapter 13: Complex Numbers

Exercise 13a

(1) $i^7 = (i^2)^3 \cdot i = (-1)^3 \cdot i = -i$

$i^{-3} = \frac{1}{i^3} = \frac{1}{i^3} \cdot \frac{i}{i} = \frac{i}{i^4} = \frac{i}{(i^2)^2} = i$

$i^9 = (i^2)^4 \cdot i = i$

$i^{-5} = \frac{1}{i^5} = \frac{1}{i^5} \cdot \frac{i}{i} = \frac{i}{(i^2)^3} = -i$

$i^{4n} = (i^4)^n = (i^2)^{2n} = 1$

$i^{4n+1} = i^{4n} \cdot i = i$

(2) $(3+5i)+(7-i) = 10+4i$; $(4-i)+(3+3i) = 7+2i$

$(2+7i)+(4-9i) = 6-2i$; $(a+bi)+(c+di)$

$= (a+c)+i(b+d)$

(3) $(3+5i)-(7-i) = -4+6i$; $(4-i)-(3+3i) = 1-4i$

$(2+7i)-(4-9i) = -2+16i$; $(a+bi)-(c+di)$

$= (a-c)+(b-d)i$

(4) (a) $(2+i)(3-4i) = 6-8i+3i-4i^2 = 10-5i$

(b) $(5+4i)(7-i) = 35-5i+28i-4i^2 = 39+23i$

(c) $(3-i)(4-i) = 12-3i-4i+i^2 = 11-7i$

(d) $(3+4i)(3-4i) = 9-16i^2 = 25$

(e) $(2-i)^2 = (2-i)(2-i) = 4-4i+i^2 = 3-4i$

(f) $(1+i)^3 = 1+3i+3i^2+i^3$ by binomial Theorem

$= 1+3i-3-i = -2+2i$

(g) $i(3+4i) = 3i+4i^2 = -4+3i$

(h) $(x+iy)(\cancel{x}-iy) = x^2-i^2y^2 = x^2+y^2$

(i) $i(1+i)(2+i) = i(2+i+2i+i^2)$

$= 2i+i^2+2i^2+i^3$

$= 2i-1-2-i$

$= -3+i$

④ $(a+ib)^2 = (a+ib)(a+ib)$

$= a^2 + 2abi + i^2b^2$

$= (a^2-b^2) + 2abi$

⑤ ⓐ $\frac{2}{1-i} = \frac{2}{1-i} \cdot \frac{1+i}{1+i}$

$= \frac{2+2i}{1-i^2} = 1+i$

ⓑ $\frac{3+i}{4-3i} = \frac{3+i}{4-3i} \cdot \frac{4+3i}{4+3i}$

$= \frac{12+9i+4i+3i^2}{16-9i^2} = \frac{9+13i}{25}$

ⓒ $\frac{4i}{4+i} = \frac{4i}{4+i} \cdot \frac{4-i}{4-i}$

$= \frac{16i-4i^2}{16-i^2} = \frac{1}{17}(4+16i)$

ⓔ $\frac{7-i}{1+7i} = \frac{7-i}{1+7i} \cdot \frac{1-7i}{1-7i}$

$= \frac{7-49i-i+7i^2}{1-49i^2}$

$= \frac{1}{50}(-50i) = -i$

(d) $\frac{1+i}{1-i} = \frac{1+i}{1-i} \cdot \frac{1+i}{1+i}$

$= \frac{1+2i+i^2}{1-i^2} = i$

(f) $\frac{x+iy}{x-iy} = \frac{x+iy}{x-iy} \cdot \frac{x+iy}{x+iy}$

$= \frac{x^2+2xyi+i^2y^2}{x^2-i^2y^2}$

$= \frac{(x^2-y^2)+2xyi}{x^2+y^2}$

(Note That if $x=y$ Then The answer Simplifies to $\frac{2x^2i}{2x^2} = i$)

(g) $\frac{3+i}{i} = \frac{3+i}{i} \cdot \frac{-i}{-i} = \frac{-3i-i^2}{-i^2} = 1-3i$

(h) $\frac{-2+3i}{-i} = \frac{-2+3i}{-i} \cdot \frac{i}{i}$

$= \frac{-2i+3i^2}{-i^2} = -3-2i$

(6) (a) $x+iy = (3+i)(2-3i)$

$= 6-9i+2i-3i^2$

$= 9-7i$

$\therefore\ x=9\ \&\ y=-7$

(b) $\dfrac{2+5i}{1-i} = x+iy$

Method 1 : $\dfrac{2+5i}{1-i} = \dfrac{2+5i}{1-i} \cdot \dfrac{1+i}{1+i}$

$= \dfrac{2+2i+5i+5i^2}{1-i^2}$

$= \dfrac{-3+7i}{2}$

$\therefore \quad x = -\frac{3}{2} \;\&\; y = \frac{7}{2}$

Method 2 : $\dfrac{2+5i}{1-i} = x+iy$

$\Rightarrow \quad 2+5i = (x+iy)(1-i)$

$= x - xi + iy - yi^2$

$= x+y+i(y-x)$

So $2 = x+y \quad \& \quad 5 = y-x$.

Solving simultaneously we get $x = -\frac{3}{2}$, $y = \frac{7}{2}$.

(c) $(3+4i) = (x+iy)(1+i)$

$= x + xi + iy + i^2 y$

$= (x-y) + i(x+y)$

So $3 = x-y \quad \& \quad 4 = x+y$

Solving simultaneously gives $x = \frac{7}{2}$, $y = \frac{1}{2}$

we could Also have solved This way

$$\frac{3+4i}{1+i} = x+iy$$

so $\frac{3+4i}{1+i} = \frac{3+4i}{1+i} \cdot \frac{1-i}{1-i} = \frac{3-3i+4i-4i^2}{1-i^2}$

$$= \frac{7+i}{2}$$

so $x = \frac{7}{2}$, $y = \frac{1}{2}$

(d) $x+iy = 2 \Rightarrow x+iy = 2+0i$

$\therefore x = 2, y = 0$

(e) $x+iy = (3+2i)(3-2i)$

$= 9 - 4i^2 = 13$

$= 13 + 0i$

$\therefore x = 13 , y = 0$

(f) $x+iy = (4+i)^2 = 16+8i+i^2 = 15+8i$

so $x = 15, y = 8$

(g) $\dfrac{x+iy}{2+i} = 5-i$

LHS: $\dfrac{x+iy}{2+i} = \dfrac{x+iy}{2+i} \cdot \dfrac{2-i}{2-i}$

$$= \frac{2x - ix + 2iy - i^2y}{4-i^2}$$

$$= \frac{2x+y + i(2y-x)}{5}$$

So $\dfrac{2x+y}{5} = 5$ & $\dfrac{2y-x}{5} = -1$

Solve simultaneously to get $x=11$, $y=3$. We could also have solved by doing $(x+iy) = (5-i)(2+i)$ ....

(h) $(x+iy)^2 = 3+4i$

$x^2 + 2xyi + i^2y^2 = 3+4i$

So $x^2 - y^2 = 3$ & $2xy = 4$

$xy = 2$

$y = \dfrac{2}{x}$

$\therefore \quad x^2 - \left(\dfrac{2}{x}\right)^2 = 3 \Rightarrow x^4 - 4 = 3x^2$

$\therefore \quad x^4 - 3x^2 - 4 = 0$

This is a quadratic in x^2: $(x^2)^2 - 3(x)^2 - 4 = 0$
So use formula on x^2

$$x^2 = \frac{+3 \pm \sqrt{9+4(4)}}{2} = 4, -1$$

$x^2 = -1$ is Not a valid Solution since x is a Real No.

So $x^2 = 4 \Rightarrow x = \pm 2$.

From $y = \frac{2}{x}$ we get $y = \pm 1$

we Now need to test These Values to see which set is correct.
So

$$(2+i)^2 = 4 + 4i + i^2 = 3 + 4i$$

&

$$(-2-i)^2 = 4 + 4i + i^2 = 3 + 4i$$

So both $x = 2, y = 1$ & $x = -2, y = -1$ work

~~[illegible]~~

(7) (a) $(2-i)(3+i) = 6 + 2i - 3i - i^2 = 7 - i$

So Real part $= 7$, imaginary part $= -1$ (The ans in The book is Not correct)

1st See comment at End of Solution (7).

(b) $(1-i)^3 = 1 - 3i + 3i^2 - i^3$

$$= -2 - 3i - i \cdot i^2$$

$$= -2 - 2i$$

So $\mathrm{Re}\,(1-i)^3 = -2$

$\mathrm{Im}\,(1-i)^3 = -2$ (The answer in The book is Not correct. In general $\mathrm{Im}\,(x+iy) = y$ Not iy)

(c) $$\frac{3+2i}{4-i} = \frac{3+2i}{4-i} \cdot \frac{4+i}{4+i}$$

$$= \frac{12+3i+8i+2i^2}{16-i^2} = \frac{10+11i}{17}$$

So $$\mathrm{Re}\left(\frac{3+2i}{4-i}\right) = \frac{10}{17}$$

$$\mathrm{Im}\left(\frac{3+2i}{4-i}\right) = \frac{11}{17}$$

(The answer in the book is Not correct)

(d) $$\frac{2}{3+i} + \frac{3}{2+i} = \frac{2}{3+i} \cdot \frac{3-i}{3-i} + \frac{3}{2+i} \cdot \frac{2-i}{2-i}$$

$$= \frac{6-2i}{10} + \frac{6-3i}{5}$$

$$= \frac{6-2i}{10} + \frac{12-6i}{10}$$

$$= \frac{18-8i}{10} = \frac{9-4i}{5}$$

So $$\mathrm{Re}\left(\frac{2}{3+i} + \frac{3}{2+i}\right) = \frac{9}{5}$$

$$\mathrm{Im}\left(\frac{2}{3+i} + \frac{3}{2+i}\right) = \frac{-4}{5}$$

(The answer in The book is Not correct)

(e) $$\frac{1}{x+iy} - \frac{1}{x-iy} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy}$$

$$- \frac{1}{x-iy} \cdot \frac{x+iy}{x+iy}$$

$$\frac{1}{x+iy} - \frac{1}{x-iy} = \frac{x-iy}{x^2+y^2} - \frac{x+iy}{x^2+y^2}$$

$$= \frac{-2iy}{x^2+y^2}$$

So Real part is 0, and Im part is $-2y/(x^2+y^2)$

(f) $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^3 = \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^3$

By Binomial Theorem:

$$\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^3 = \left(\frac{\sqrt{3}}{2}\right)^3 + 3\left(\frac{\sqrt{3}}{2}\right)^2\left(\frac{1}{2}i\right) + 3\frac{\sqrt{3}}{2}\left(\frac{1}{2}i\right)^2 + \left(\frac{1}{2}i\right)^3$$

$$\frac{1}{x+iy} - \frac{1}{x-iy} = \frac{x-iy}{x^2+y^2} - \frac{x+iy}{x^2+y^2}$$

$$= \frac{2iy}{x^2+y^2}$$

So Real part is 0 and Im part is $-\frac{2y}{x^2+y^2}$

(f) $\left(\cos\frac{\pi}{3} + i.\sin\frac{\pi}{3}\right)^3 = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3$

By the binomial theorem

$$\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3 = \left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2\left(\frac{\sqrt{3}}{2}i\right) + 3\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}i\right)^2 + \left(\frac{\sqrt{3}}{2}i\right)^3$$

Looking at The Real parts: $\left(\frac{1}{2}\right)^3 + \frac{3}{2}\left(\frac{\sqrt{3}}{2}i\right)^2 = \frac{1}{8} - \frac{9}{8} = -1$

" " Im " : $\frac{3}{4}.\frac{\sqrt{3}}{2}i + \left(\frac{\sqrt{3}}{2}\right)^3 i^3$

$$= \frac{3\sqrt{3}}{8}i + \frac{(\sqrt{3})^2\sqrt{3}}{8}i^3$$

$$= \frac{3\sqrt{3}}{8}i - \frac{3\sqrt{3}}{8}i = 0$$

(g) $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^2 = \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^2$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 + 2.\frac{1}{2}.\frac{\sqrt{3}}{2}i + \frac{1}{4}i^2$$

$$= \left(\frac{3}{4} - \frac{1}{4}\right) + \frac{\sqrt{3}}{2}i$$

So Real part is $\frac{1}{2}$ and Im part is $\frac{\sqrt{3}}{2}$

$\Longrightarrow$

Note : All The answers in The book for question 7 are technically incorrect. The Numerical Values are correct but ...

... For $x+iy$ The Real part is x
The Im part is y (Not iy)

8 a $z = 3-4i$

let $\sqrt{3-4i} = a+ib$

So $3-4i = (a+ib)^2 = (a^2-b^2) + 2iab$

So Re : $3 = a^2 - b^2$
Im : $-4 = 2ab \Rightarrow b = -\frac{2}{a}$

$\therefore \quad 3 = a^2 - \left(-\frac{2}{a}\right)^2 = a^2 - \frac{4}{a^2}$

$\therefore \quad 3a^2 = a^4 - 4$

$\therefore \quad a^4 - 3a^2 - 4 = 0$

Solving as a quadratic in a^2 we get

$$(a^2-4)(a^2+1) = 0$$

$\therefore \quad a^2 = 4 \quad$ or $\quad a^2 = -1$

But a is a Real N° so $a^2 = -1$ is Not Valid.
Hence

$$a^2 = 4 \Rightarrow a = \pm 2 \Rightarrow b = \mp 1$$

$\therefore \quad \sqrt{3-4i} = 2-i, \; -2+i$

(b) $Z = 21 - 20i$

let $\sqrt{21-20i} = a + ib$

So $21 - 20i = (a+ib)^2$

$= a^2 - b^2 + 2abi$

$\therefore$ Re : $21 = a^2 - b^2$

Im : $-20 = 2ab \Rightarrow b = -\frac{10}{a}$

hence $21 = a^2 - \left(-\frac{10}{a}\right)^2$

$\Rightarrow 21a^2 = a^4 - 100$

So $a^4 - 21a^2 - 100 = 0$

Solve as a quadratic in a^2: $(a^2 - 25)(a^2 + 4) = 0$

So $a^2 = 25 \Rightarrow a = \pm 25$

and $a^2 = -4 \Rightarrow a = \pm 2i$

But a is Real So $a = \pm 5$

and $\therefore b = \mp 2$

$\therefore \sqrt{21-20i} = \pm(5 - 2i)$

(c) $z = 2i$

let $\sqrt{2i} = a + ib$

So $2i = (a+ib)^2 = (a^2 - b^2) + 2abi$

Re: $0 = a^2 - b^2 \Rightarrow a^2 = b^2 \Rightarrow a = \pm b$

Im: $2 = 2ab \Rightarrow b = \frac{1}{a}$

hence $a^2 = \left(\frac{1}{a}\right)^2$

$\therefore a^4 = 1 \Rightarrow (a^4 - 1) = 0$

$\therefore (a^2 - 1)(a^2 + 1) = 0$

& $\therefore (a-1)(a+1)(a^2+1) = 0$

Hence $a = 1, -1, \pm i$

But a is Real, $\therefore a = \pm 1$ & This implies $b = \pm 1$

$\therefore \sqrt{2i} = \pm (1 + i)$

(d) $z = 15 + 8i$

let $\sqrt{15 + 8i} = a + ib$

$\therefore 15 + 8i = (a + ib)^2 = (a^2 - b^2) + 2abi$

Re: $15 = a^2 - b^2$

Im: $8 = 2ab \Rightarrow b = \frac{4}{a}$

$\therefore \quad 15 = a^2 - \left(\frac{4}{a}\right)^2$

& we end up with $\quad 15a^2 = a^4 - 16.$

Hence $\quad a^4 - 15a^2 - 16 = 0$

and $\therefore \quad (a^2 - 16)(a^2 + 1) = 0.$

Therefore $\quad a^2 = 16 \Rightarrow a = \pm 4$

and $\quad a^2 = -1 \Rightarrow a = \pm i$

But a is Real $\therefore$ $a = \pm 4$, From which $b = \pm 1$

$\therefore \quad \sqrt{15 + 8i} = \pm (4 + i)$

(e) $\quad \underline{z = -24 + 10i}$

Let $\quad -24 + 10i = (a + ib)^2$

$= (a^2 - b^2) + 2abi$

So $\quad$ Re : $-24 = a^2 - b^2$

Im : $\quad 10 = 2ab \Rightarrow b = \frac{5}{a}$

$\therefore \quad -24 = a^2 - \left(\frac{5}{a}\right)^2 \Rightarrow -24a^2 = a^4 - 25$

Hence $\quad a^4 + 24a^2 - 25 = 0$

$\therefore \quad (a^2 + 25)(a^2 - 1) = 0$

and $\therefore \quad a = \pm 5i$ and $a = \pm 1$

But a is real so $a = \pm 1$, and $\therefore$ $b = \pm 5$

So $\sqrt{-24 + 10i} = \pm (1 + 5i)$